J Glob Optim (2008) 40:155–160
DOI 10.1007/s10898-007-9216-3

On the uniqueness of solutions of spectral equations

Yousong Luo

Received: 7 June 2007 / Accepted: 20 July 2007 / Published online: 4 August 2007

Abstract We prove uniqueness of the viscosity solutions of the Dirichlet problem of the spectral equation $F(u) = f(\lambda[u]) = \psi$ where $\lambda[u]$ is the vector whose components are eigenvalues of a matrix associated with the unknown function u.

Keywords Comparison principle · Viscosity solution · Curvature equation · Hessian equation · Spectral function

1 Introduction

We consider in this paper the Dirichlet problem of fully nonlinear partial differential equations:

$$F(Du, D^2u) = f(\lambda[u]) = \psi(x) \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, f is a function of λ and $\lambda[u]$ is the n-vector formed by the eigenvalues of the Hessian matrix D^2u or the matrix

$$M(p, A) = \frac{1}{v}PAP, \quad P = I - \frac{p \otimes p}{v(1+v)} \text{ with } v = \sqrt{1+|p|^2}, \ p = Du, \ A = D^2u. \tag{2}$$

In the later case λ is the principle curvature of the graph of u. Such a function f is generally called a spectral function, see Lewis [10] for details. Two specific examples of f are

$$f(\lambda) = S_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \text{ for } 1 \le k \le n \text{ and } S_0(\lambda) = 1, \tag{3}$$

Y. Luo (✉)
School of Mathematical and Geospatial Sciences, RMIT University, GPO Box 2476V
Melbourne, VIC 3001, Australia
e-mail: yluo@rmit.edu.au

and

$$f(\lambda) = S_{k,l}(\lambda) = \frac{S_k(\lambda)}{S_l(\lambda)} \quad \text{for } 0 \le l < k \le n. \tag{4}$$

The classical existence and uniqueness of such a problem have been extensively studied by Caffarelli et al. [2], Invochkina [5], Lin and Trudinger [9], Ivochkina et al. [7] among other authors. The existence and uniqueness of weak solutions in the viscosity sense of Crandall and Lions [3] have also been well studied by Trudinger and some other authors. Urbas [15] has also proved a non-existence theorem for the classical solutions of such a problem. The uniqueness of the viscosity solutions of a general fully nonlinear equation was first obtained by Jensen [8] but the conditions there are not satisfied by either Hessian equations or curvature equations. Trudinger [13] proved a uniqueness result for (1) in the case $\inf_\Omega \psi > 0$. The uniqueness in the complete degenerate case $\psi \equiv 0$ was published by Cranny [4] but there was not complete proof. The rigorous proof of this result was done by the author [12] later. The remaining case is the general degenerate case $\psi \ge 0$ where ψ is allowed to be positive somewhere and zero otherwise. This problem is till open and the difficulty in answering it is due to the lack of strong ellipticity and the absence of u variable in the function F which are the key ingredients in establishing a comparison principle. Also the presence of x always causes trouble in proving a comparison principle. We give an answer to this problem here to bring this old question to an end.

To formulate the appropriate notion of weak solution, we define admissible cone Γ of the function f as a convex, connected cone, containing the positive cone and contained in the half space $\sum_{i=1}^{n} \lambda_i \ge 0$ such that

$$\Gamma = \{\lambda \in \mathbb{R}^n \mid f(\lambda) \ge 0\}.$$

The existence of such a cone is guaranteed if f is concave and $f(0) = 0$. We assume that f is a continuous concave function satisfying $f(0) = 0$ and the following conditions:

f1 $f(\lambda + \mu) \ge f(\lambda)$ for all $\mu \ge 0$ and $f(\lambda + \mu) > f(\lambda)$ for all $\mu > 0$, provided $\lambda \in \Gamma$.
f2 $f(t\lambda) \to \infty$ as $t \to \infty$ for any $\lambda \in \Gamma$.
f3 Over smooth domain Ω and for $\psi \in C^2$ and smooth g the problem

$$f(\lambda[u]) = \psi \ \text{ in } \ \Omega, \quad u = g \ \text{ on } \ \partial\Omega$$

has classical solution.
f4 The solution u of the equation $f(\lambda[u]) = \psi$ in Ω has the a priori bound

$$\sup_{\Omega'} |D^2 u| \le C$$

where Ω' is a compact sub domain strictly contained in Ω and C depends only on ψ and $\mathrm{dist}(\Omega', \partial\Omega)$.

The conditions f3 and f4 are derived from various conditions of the classical existence theory. The detailed conditions for Hessian and curvature equations can be found in the works of Trudinger, Caffarelli, Nirenberger, Spruck Invochkina ans so on mentioned earlier.

The notion of viscosity sub-solution here is the standard one of Crandall and Lions [3], that is, a USC function u is a viscosity sub-solution of (1) if $u \le g$ on $\partial\Omega$ and for any $x_0 \in \Omega$ and $(p, A) \in \partial^{2,+}u(x_0)$ we have $F(p, A) \ge \psi(x_0)$. However for super-solutions we adopt Trudinger's definition, that is, a LSC function u is a viscosity super-solution of (1) if $u \ge g$ on $\partial\Omega$ and for any $x_0 \in \Omega$ any admissible $(p, A) \in \partial^{2,-}u(x_0)$ we have $F(p, A) \le \psi(x_0)$. Here a pair $(p, A) \in \mathbb{R}^n \times \mathcal{S}(n)$ is called admissible if the eigenvalues λ of A in the Hessian equation case or of $M(p, A)$ in the curvature equation case belongs to $\overline{\Gamma}$ where $M(p, A)$ is defined by (2). A function u is a viscosity solution if and only if it is both sub- and super-solution.

Theorem 1 *Suppose that $\psi \geq 0$ is in $C^2(\overline{\Omega})$, g is Lipschitz continuous and f satisfies the conditions f1–f4. If u is a continuous sub-solution and v is a continuous super-solution of the problem* (1) *then $u \leq v$ in Ω.*

2 Sketch of the Proof of the Theorem

The following result of the author (see [11]) plays the crucial roll in the argument.

Lemma 1 *Let u be an upper semi-continuous function over a bounded domain Ω of $\mathbb{R}^n$. Assume that argmax $(u) \subseteq$ intΩ. Then for any $\varepsilon > 0$ and $\delta > 0$ there is an open subset $U \subset \Omega$ and a C^∞ function φ such that $u - \varphi$ achieves its maximum on U together with, for each $x \in U$,*

1 $\mathrm{dist}(x, \mathit{argmax}(u)) < \delta\varepsilon$ *and* $\mathrm{diam}(U) \leq \delta\varepsilon$;
2 $A = D^2\varphi(x)$ *is a negative definite matrix whose eigenvalues κ_i satisfy* $0 > \kappa_i > -\varepsilon$ *for* $i = 1, 2, \ldots, n$;
3 $p = D\varphi(x)$ *satisfies* $|p| < \varepsilon \min_i |\kappa_i|$.

From the proof of this lemma in [11] one can see that for any sequence $\{u_n\}$ uniformly converging to u the function φ and hence the magnitude of κ_i can be chosen to be independent of n.

The following result (see [12]) about the continuity of the matrix (2) is also needed here.

Lemma 2 *If $(q_1, A_1), (q_2, A_2) \in \mathbb{R}^n \times \mathcal{S}(n)$ satisfy*

$$|q_1|, \quad |q_2| \leq k_1 \quad \text{and} \quad -k_2 I \leq A_1 \leq A_2 - \sigma I \leq k_2 I \tag{5}$$

for some positive constants k_1, k_2 and σ, then there are positive constants δ and c_1 dependent only on k_1 and k_2 such that

$$M(q_1, A_1) < M(q_2, A_2) - c_1 \sigma I$$

whenever

$$|q_1 - q_2| \leq \delta\sigma. \tag{6}$$

Let us establish the comparison principle for the case that sub- or super-solution is in C^2.

Lemma 3 *Assume that $f(\lambda)$ satisfies conditions f1 and f2 and λ is the principle curvature. If u is a continuous sub-solution and v is a continuous super-solution of the equation*

$$F(Du, D^2u) = f(\lambda[u]) = \psi(x) \quad \text{in } \Omega$$

and either u or v is in $C^2(\Omega)$ then $u - v$ can not achieve a strict maximum in Ω.

Proof Suppose the contrary that $u - v$ achieves a strict maximum at some interior point of Ω. Assume $v \in C^2$. Applying Lemma 1 to the function $u - v$ we have $(0, 0) \in \partial^{2,+}(u - v - \varphi)(x_0))$ at a maximum point x_0 of $u - v - \varphi$ in the interior of Ω. Since both v and φ are in C^2 we have $\partial^{2,+}u(x_0) \neq \emptyset$. Let $(p, A) \in \partial^{2,+}u(x_0)$ and $(q, B) = (Dv(x_0), D^2v(x_0))$.

Then $p = q + D\varphi(x_0)$, $A \le B + D^2\varphi(x_0) \le B - \sigma I$ with $\sigma = \min\{|\kappa_i|\}$ and (p, A) is admissible. It is obvious that A is also bounded below by $-KI$ for a constant K dependent only on $|D^2 v|$ because otherwise the negative eigenvalue of A with sufficient large magnitude will drag (p, A) out of the admissible cone. With the ε in Lemma 1 sufficiently small we have $|p - q| = |D\varphi(x_0)| < \varepsilon < \delta\sigma$ for the constant δ given in Lemma 2 which is dependent only on K. Then by Lemma 2 we have a positive c_1 dependent only on K such that $M(p, A) < M(q, B) - c_1\sigma I$. It follows from condition f1 that

$$\psi(x_0) \le F(p, A) < F(q, B) \le \psi(x_0)$$

a contradiction.

Let us now consider the case that u is in C^2, then $\partial^{2,-}v(x_0) \ne \emptyset$. Taking any $(q, B) \in \partial^{2,-}v(x_0)$ we will have $D^2u(x_0) \le B + D^2\varphi(x_0) \le B - \sigma I$. This implies that B is bounded from below. It is also bounded from above because of condition f2 and $F(q, B) \le \psi(x_0)$. The remaining proof is the same as above. □

Corollary 1 *The result of Lemma 3 holds true if λ is the eigenvalues of the Hessian matrix of u.*

Now we turn to the proof of the Theorem 1.

Proof Assume the contrary that $u - v$ achieves a positive maximum at some interior point of Ω. Applying Lemma 1 to the function $u - v$ we have a small open subset $U \subset \Omega$ and a C^∞ function φ such that $u - v - \varphi$ achieves its maximum in U. Without loss of generality we can assume that U is a ball. Let $\overline{x}$ be any point in U where $u - v - \varphi$ achieves its maximum. Then we have

$$(u - v - \varphi)|_{\partial U} < (u - v - \varphi)(\overline{x}). \tag{7}$$

Now we slightly lift u and then mollify it to get a smooth u_h such that $u_h \ge u$ on ∂U and (7) still holds with u replaced by u_h on the left-hand-side. By the assumption f3 there is a classical solution u_1 to the problem

$$f(\lambda[u_1]) = \psi \;\text{ in } U, \quad u_1 = u_h \;\text{ on } \partial U.$$

Since the classical solution u_1 is also a super-solution we conclude, by Lemma 3, that $u \le u_1$ in U. It follows that

$$(u_1 - v - \varphi)|_{\partial U} = (u_h - v - \varphi)|_{\partial U} < (u - v - \varphi)(\overline{x}) < (u_1 - v - \varphi)(\overline{x}). \tag{8}$$

This implies that $(u_1 - v - \varphi)$ achieves its maximum in the interior of U. The remaining proof is similar to that of Lemma 3. We only discuss the curvature case. We may further assume that $(u_1 - v - \varphi)$ achieves its maximum in $U_1 \subset U$ with $d = \operatorname{dist}(U_1, \partial U) > 0$. Let $x_0 \in U_1$ be a maximum point of $(u_1 - v - \varphi)$. Then we have $\partial^{2,-}v(x_0) \ne \emptyset$ so there is a jet $(q, B) \in \partial^{2,-}v(x_0)$ and $(p, A) = (Du_1(x_0), D^2u_1(x_0))$ such that $p = q + D\varphi(x_0)$, $A \le B + D^2\varphi(x_0) \le B - \sigma I$. By assumption f4 we have a bound for $|D^2u_1|$ in U_1. In terms of this bound we have lower bound for B. This implies that all the negative components of B are bounded. If some of the positive components of B is unbounded then by f2 we will have a contradiction to $f(\lambda[B]) \le \psi$. In this way, we obtain a bound for B in terms of $|D^2u_1|$. With this bound we can now apply Lemma 2 as before to get $M(p, A) < M(q, B) - c_1\sigma I$ provided $|p - q| \le \delta\sigma$ for a small constant δ dependent only on $\sup_{U_1} |D^2u_h|$. Recall that $p - q = D\varphi(x_0)$ so $|p - q| \le \delta\sigma$ can be achieved if ε in Lemma 1 is chosen to be less than δ. From the proof of Lemma 1 we can see that, to decrease ε we should flatten φ by decreasing the parameter in it. Doing this we will only shrink U to get a new $U' \subset U$ so this will not affect the argument above. Finally we have

$$\psi(x_0) = F(p, A) < F(q, B) \leq \psi(x_0) \tag{9}$$

and we get the conclusion because this is clearly a contradiction. □

3 Further remarks

3.1 Assumption on classical existence

The assumption f3 on existence of classical solution is almost automatically satisfied in most of cases. For example, in the case of quotient curvature equation the following is the summary of such a result by Ivochkina [6].

Theorem 2 *Assume that*

1 $2R = \operatorname{diam}\Omega < \infty, \Omega \in C^4, S_m(\partial\Omega) > 0, g \in C^4(\partial\Omega)$*;*
2 $0 < \psi < 1/R$ *in* $\overline{\Omega}, \psi \in C^2(\overline{\Omega})$*, and for* $x \in \partial\Omega$ $\psi \leq S_{m,l}(\partial\Omega)$.

Then there exists a solution $u \in C^{3+\alpha}(\overline{\Omega}), 0 < \alpha < 1$*, to problem* (1).

Applying this Theorem in our proof the conditions are automatically met. First, our Ω is a small ball so $S_m(\partial\Omega) > 0$. Second, when $\psi \in C^2(\overline{\Omega})$ we can always choose our ball U so small that the curvature of the boundary is large enough. In this way, we will have $\psi < 1/R$ in $\overline{U}$ and for $x \in \partial U$ $\psi \leq S_{m,l}(\partial U)$. To fulfill the requirement of $\psi > 0$ we can use $\psi + a$ with a small positive constant in the construction of u_h. In this end, we will have (9) in the form

$$\psi(x_0) + a = F(p, A) < F(q, B) \leq \psi(x_0)$$

which still produces a contradiction when a is small enough.

3.2 Assumption on interior estimates

The interior estimates of type f4 in various cases can be found in [14] and references wherein.

3.3 More general equations

We suppose that the result is still true for the more general class $\psi = \psi(x, u, Du)$ because the argument carries over to such a case without any significant change, as long as ψ is monotonically non-decreasing in u. We will summarize the existence result for f3 and the a priori estimate result for f4 in details in a in a forthcoming study.

3.4 Admissible cone

Our proof for the uniqueness works well also for some other class of equations where the notion of admissible solution is not needed, i.e., we do not have to restrict λ within the admissible cone. One example of such equations is the prescribed mean curvature equation which, in our setting, takes the form

$$f(\lambda) = \sum \lambda_i = \psi.$$

This equation is not included in the general case of the curvature quotient equation because $f(\lambda)$ is not concave. The classical theory of this equation has been thoroughly studied in literature but there are still new works on weak solutions, e.g., Amester et al. [1] where existence and multiplicity of weak solutions in the distribution sense are obtained. Our result shows that the viscosity solution of such equation is unique. The conditions f3 and f4 are always satisfied over small balls. The additional assumption replacing the admissibility is that if all the positive components of λ are bounded from above then $\lim_{|\lambda|\to\infty} f(\lambda) < \min_{\Omega} \psi$.

Acknowledgement This research is supported by the ARC grant number DP0451168.

References

1. Amster, P., Cassinelli, M., Mariani, M.C., Rial, D.F.: Existence and regularity of weak solutions to the prescribed mean curvature equation for a nonparametric surface. Abstr. Appl. Anal. **4**(1), 61–69 (1999)
2. Caffarelli, L., Nirenberg, L., Spruck, J.: Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces. Comm. Pure Appl. Math. **41**(1), 47–70 (1988)
3. Crandall, M.G., Lions, P.-L.: Viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc. **277**(1), 1–42 (1983)
4. Cranny, T.R.: On the uniqueness of solutions of the homogeneous curvature equations. Ann. Inst. H. Poincar Anal. Non Linaire **13**(5), 619–630 (1996)
5. Ivochkina, N.M.: Solution of the Dirichlet problem for an equation of curvature of order m. (Russian) Dokl. Akad. Nauk SSSR **299**(1), 35–38 (1988); Translation in Soviet Math. Dokl. **37**(2), 322–325 (1988)
6. Ivochkina, N.M.: Local estimates of normal second derivatives for solutions of the Dirichlet problem for the prescribed quotient curvature equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 221 (1995), Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. **26**, 114–126, 257; Translation in J. Math. Sci. (New York) **87**(2),3343–3352 (1997)
7. Ivochkina, N., Trudinger, N., Wang, X.-J.: The Dirichlet problem for degenerate Hessian equations. Comm. Partial Diff. Eqs. **29**(1–2), 219–235 (2004)
8. Jensen, R.: The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. Arch. Rat. Mech. Anal. **101**, 1–27 (1988)
9. Lin, M., Trudinger, N.S.: The Dirichlet problem for the prescribed curvature quotient equations. Topol. Method Nonlinear Anal. **3**(2), 307–323 (1994)
10. Lewis, A., Sendov, H.: Twice differentiable spectral functions. SIAM J. Matrix Anal. Appl. **23**(2), 368–386 (2001)
11. Luo, Y., Eberhard, A.: Comparison principles for viscosity solutions of elliptic equations via Fuzzy Sum Rule. J. Math. Anal. Appl. **307**(2), 736–752 (2005)
12. Luo, Y., Eberhard, A.: An application of $C^{1,1}$ approximation to comparison principles for viscosity solutions of curvature equations. Nonlinear Anal. **64**(6), 1236–1254 (2006)
13. Trudinger, N.S.: The Dirichlet problem for the prescribed curvature equations. Arch. Rat. Mech. Anal. **111**(1), 153–179 (1990)
14. Trudinger, N.S.: On the Dirichlet problem for Hessian equations. Acta Math. **175**(2), 151–164 (1995)
15. Urbas, J.I.E.: On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations. Indiana Univ. Math. J. **39**(2), 355–382 (1990)